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Creating and Transforming a Second-Rank Antisymmetric Field-Strength Tensor $F^{\alpha\beta}$ in Minkowski Space using MATHEMATICA

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As the laws of physics are expressed in a manner that makes their invariance under coordinate transformations manifest, they should be written in terms of tensors. Furthermore, tensors make manifest the characteristics and behaviors of electromagnetic fields through inhomogeneous, anisotropic, and compressible media. Electromagnetic fields are expressed completely in tensor form, $F^{\alpha\beta}$, which implies both electric field $\vec{E}$ and magnetic field $\vec{B}$ rather than separately in the vector fields. This study presents the Mathematica platform that generates and transforms a second-rank antisymmetric field-strength tensor $F^{\alpha\beta}$ and whiskbroom pattern in Minkowski space. The platforms enhance the capabilities of students and researchers in tensor analysis and improves comprehension of the elegant features of complete structure in physics.

Keywords: covariant field tensor, MATHEMATICA platform, whiskbroom platform, second-rank antisymmetric field-strength tensor $F^{\alpha\beta}$, Minkowski space

1. INTRODUCTION

Vectors and tensors are the most fundamental study subjects both in undergraduate and graduate physics courses (Marion 1970; Jackson 1975; Fowles 1975; Griffiths 1981; Reitz 1993), as the laws of physics are expressed in a manner that makes their invariance, under coordinate transformation, manifest the characteristics and behaviors of electromagnetic fields in inhomogeneous, anisotropic, and compressible media. The electromagnetic fields are expressed completely in a *second-rank antisymmetric field-strength tensor*, $F^{\alpha\beta}$, which implies both electric field $\vec{E}$ and magnetic field $\vec{B}$, rather than separately in the vector fields. Then, electric and magnetic fields, $\vec{E}$ and $\vec{B}$ respectively are the elements of a field tensor $F^{\alpha\beta} = \partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha}$, with the contravariant differential operator ∂^{α} and the contravariant 4-vector $A^{\alpha} = \left(A^0, \vec{A}\right)$ in Minkowski space instead of the vector differential operator $\vec{\nabla}$ and $\vec{A}$ in 3-vector space (Arfken 1970; Flügge 1972; Georgi 1982; Parker 1994). Once $F^{\alpha\beta}$ is obtained, we can get the covariant form $F_{\alpha\beta}$ of this tensor using

metric tensor $g_{\alpha\gamma}g_{\delta\beta}$ and $F'^{\mu\nu}$ coordinate transformed by the Lorentz transformation in Minkowski space (Fitzpatrick 2019; Yamamoto 2019). As interested in the quantum cryptography of secure quantum communications has received much attention in recent year, we must examine carefully quantum entanglement (Ekhert 1991; Fedrizzi 2013); new artificial metamaterials ($\mu\varepsilon\tau\alpha'$ in Greek) are controlled by the ε permittivity tensor and μ permeability tensor (Veselago 2006; Kwoen 2018). For the geometry of spacetime in cosmology, Einstein field equation, $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$, is expressed with the Einstein tensor $G_{\mu\nu}$ of the curvature of spacetime and stress-energy tensor $T_{\mu\nu}$ with metric tensor $g_{\mu\nu}$. The Einstein's field equation is described in tensor form completely (Zimmerman 1995; Moskowitz 2017). We can see that new technology is developed fundamentally from a basis in basic science (Ekhert 1991; Veselago 2006; Fedrizzi 2013; Kim 2014; Kwoen 2018). Education environments in this age work poorly for students in physics to examine the tools of basic science. We presented the capability of MATHEMATICA for the simulations of electromagnetic

field of polarization mode (tensor of rank 1, Choi 2015) and manipulating antisymmetric field tensor $F^{\alpha\beta}$ in Minkowski space (Yun 2006; Yun 2020a; Yun 2020b), which can be utilized to help students studying the electromagnetic field tensor in Minkowski space.

In this paper, we will manifest the electromagnetic field tensor $F^{\alpha\beta}$ in Minkowski space in the MATHEMATICA platform to help readers comprehend and observe the whiskbroom patterns of the tensor field at various viewpoints. For those readers unfamiliar with tensor, we shall briefly review its basic concepts in Section II, manifest tensor, and simulate the electric fields boosted in Minkowski space with the MATHEMATICA platform; we will show *whiskbroom* pattern in Section III, and we conclude in Section IV. Some parts of the MATHEMATICA code presenting the process of generating and manipulating tensors are also included.

2. ELECTROMAGNETIC FIELD TENSOR

2.1 Second-Rank Antisymmetric Field-Strength Tensor $F^{\alpha\beta}$

From the first postulate of special relativity, the mathematical equations expressing the laws of nature must be covariant, that is, they are of the invariant form under the transformations of the Lorentz group (Fitzpatrick 2019; Perez 2019; Yamamoto 2019). The space-time continuum, also known as Minkowski space, is defined as a four-dimensional space-time with the coordinates (x^0, x^1, x^2, x^3) of which the metric $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$ (Arfken 1970; Georgi 1982; Ahn 1993). We suppose that there is a well-defined transformation that yields new the coordinates (x'^0, x'^1, x'^2, x'^3), according to some rule:

$$x'^{\alpha} = x'^{\alpha}(x^0, x^1, x^2, x^3), \alpha = 0, 1, 2, 3 \tag{1}$$

The tensors of rank k associated with the space-time point x are defined by their transformation properties under the transformation $x \to x'$. A scalar (tensor of rank zero) is a single quantity whose value is not changed by the transformation. For tensors of rank one, vectors, there are two kinds of vectors: a contravariant vector A^{α} with the components A^0, A^1, A^2, A^3 and a covariant vector B_{β} with the components B_0, B_1, B_2, B_3. Contravariant vector A^{α} and covariant vector B_{β} are transformed according to following rules respectively (Jackson 1975):

$$A'^{\alpha} = \sum_{\beta=0}^{3} \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta},$$
$$= \left(\frac{\partial x'^{\alpha}}{\partial x^0} A^0 + \frac{\partial x'^{\alpha}}{\partial x^1} A^1 + \frac{\partial x'^{\alpha}}{\partial x^2} A^2 + \frac{\partial x'^{\alpha}}{\partial x^3} A^3 \right), \tag{2}$$

$$B'_{\alpha} = \sum_{\beta=0}^{3} \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta},$$
$$= \left(\frac{\partial x^0}{\partial x'^{\alpha}} B_0 + \frac{\partial x^1}{\partial x'^{\alpha}} B_1 + \frac{\partial x^2}{\partial x'^{\alpha}} B_2 + \frac{\partial x^3}{\partial x'^{\alpha}} B_3 \right). \tag{3}$$

We will henceforth use the summation convention for repeated indices. The inner or scalar product of two vectors is defined as the product of the components of a covariant and a contravariant vector:

$$B{\cdot}A = B_{\alpha}A^{\alpha} \tag{4}$$

With this definition, the scalar product is an invariant or scalar under the transformation, Eq. (1). The scalar product $B'{\cdot}A'$ using Eqs. (2)–(3) is

$$B'{\cdot}A' = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} B_{\beta} A^{\gamma}$$
$$= \frac{\partial x^{\beta}}{\partial x^{\gamma}} B_{\beta} A^{\gamma} = \delta_{\gamma}^{\beta} B_{\beta} A^{\gamma} = B_{\beta} A^{\gamma} = B \cdot A \tag{5}$$

Thus, confirm that the scalar product is an invariant under the transformation of eq. (1). The Lorentz transformations of the four-dimensional coordinates, $(x_0, \vec{x})$ follow the invariance of

$$s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{6}$$

Here, we consider the group of all transformations that leave s^2 invariant, which is the Lorentz scalar. This group is called the *Lorentz group* (Fowles 1975; Cabrera 2019; Perez 2019). It contains ordinary rotations, as well as boosted Lorentz transformations. In differential form, the infinitesimal interval ds that defines the norm of our space is

$$(ds)^2 = (ds^0)^2 - (ds^1)^2 - (ds^2)^2 - (ds^3)^2 \tag{7}$$

This *norm* or *metric* is a special case of the general differential length element,

$$(ds)^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta} \tag{8}$$

Where, $g_{\alpha\beta}$ is called the *metric tensor* (Arfken 1970; Georgi 1982; Parker 1994). Comparison of the invariant length $(ds)^2$ in eq. (8) with the similarly invariant scalar product in eq. (6) suggests that the covariant coordinate 4-vector x_{α} can be obtained from the contravariant x^{β} by contraction with $g_{\alpha\beta}$ and its inverse x^{α} with $g^{\alpha\beta}$, that is,

$$x_{\alpha} = g_{\alpha\beta} x^{\beta}, x^{\alpha} = g^{\alpha\beta} x_{\beta} \tag{9}$$

For the flat space-time of special relativity, the metric tensor is diagonal, with elements $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$. One 4-potential is defined as $A^\alpha = (A^0, \vec{A})$ and $A_\alpha = (A^0, -\vec{A})$. The scalar product of two 4-vectors is

$$\mathbf{B}\cdot\mathbf{A} \equiv B_\alpha A^\alpha = B^0 A^0 - \vec{B}\cdot\vec{A} \qquad (10)$$

Now, we consider the partial derivative operators with respect to x^α and x_α. The transformation properties of these operators can be established directly by using the rules of implicit differentiation. For example, we have compared with the coordinate vector transformation of the covariant vector B'_α as below:

$$\frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta} \sim B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha}B_\beta$$

It shows that differentiation with respect to a contravariant component of the coordinate vector transforms as the components of a covariant vector operator. We use the notations ∂_α and ∂^α for the covariant and contravariant differential operators respectively (Jackson 1975):

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = (\partial_0, \partial_1, \partial_2, \partial_3)$$
$$= \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right) \qquad (11)$$

$$\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = (\partial^0, \partial^1, \partial^2, \partial^3)$$
$$= \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right) \qquad (12)$$

This is the 4-gradient of the 4-potential $(\varphi, \vec{A})$. The 4-divergence of 4-vector A is the following invariant:

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla}\cdot\vec{A} \qquad (13)$$

The four-dimensional Laplacian operator is defined to be an invariant contraction by Eqs. (12)–(13);

$$\Box \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{0^2}} - \nabla^2 \qquad (14)$$

Electric and magnetic fields are expressed in terms of the potentials in vector differential mode as;

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi, \ \vec{B} = \vec{\nabla}\times\vec{A} \qquad (15)$$

Then the components of these fields are expressed in terms of the 4-gradient ∂^α of four-potential A^α explicitly

$$E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \varphi}{\partial x} = -(\partial^0 A^1 - \partial^1 A^0),$$
$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2) \qquad (16)$$

where the second terms from the assignments are $x_0 = ct$, $A^0 = \varphi$, and ∂^α from Eq. (13). Now we can write the 3-vector fields of rank one to a 4-vector field tensor of rank two as;

$$\vec{E} \mapsto E^{\alpha0} = \frac{\partial A^0}{\partial x_\alpha} - \frac{\partial A^\alpha}{\partial x_0} = \partial^\alpha A^0 - \partial^0 A^\alpha \qquad (17)$$

$$\vec{B} \mapsto B^{\alpha\beta} = -\left(\frac{\partial A^\beta}{\partial x_\alpha} - \frac{\partial A^\alpha}{\partial x_\beta}\right) = -(\partial^\alpha A^\beta - \partial^\beta A^\alpha) \qquad (18)$$

These equations imply that the electric and magnetic fields, six components in all, are the elements of a *second-rank antisymmetric field-strength tensor* $F^{\alpha\beta}$ (Arfken 1970; Georgi 1982; Parker 1994),

$$F^{\alpha\beta} = \frac{\partial A^\beta}{\partial x_\alpha} - \frac{\partial A^\alpha}{\partial x_\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \qquad (19)$$

We see that $F^{\alpha\beta}$ is an *antisymmetric* tensor, for $F^{\beta\alpha} = -F^{\alpha\beta}$. Now, we can describe the electric and magnetic fields: $\vec{E}$ and $\vec{B}$ in a 3-vector space instead, with *a second-rank anti-symmetric field strength tensor* $F^{\alpha\beta}$ in Minkowski space by the 4-gradient ∂^α of 4-potential $A^\alpha = (\varphi, \vec{A})$ completely. As shown in Eq. (16), the components of $\vec{E}$ described with the elements of the field tensor $F^{\alpha\beta}$: $E_1 = E^{10} = F^{10} = \partial^1 A^0 - \partial^0 A^1$, and $B_1 = B^{32} = F^{32} = -F^{23} = -(\partial^2 A^3 - \partial^3 A^2)$ by Eqs. (16)–(19). The *second-rank antisymmetric field-strength tensor* $F^{\alpha\beta}$ in matrix form is

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha,$$
$$= \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \qquad (20)$$

Once field tensor $F^{\alpha\beta}$ is obtained, we can get its covariant field tensor $F_{\alpha\beta}$ and dual field-strength tensor $\mathcal{F}^{\alpha\beta}$ by using the metric tensor $g_{\alpha\gamma}g_{\eta\beta}$ and the pseudo tensor $\varepsilon^{\alpha\beta\gamma\delta}$ directly, and we will transform the electromagnetic field tensors with the Lorentz transformation (Georgi 1982; Parker 1994). The manipulation of these tensors is performed in MATHEMATICA conveniently (Zimmerman 1995; Yun 2006; Vrbik 2012).

3. ELECTROMAGNETIC FIELD TENSOR $F^{\alpha\beta}$ WITH MATHEMATICA

3.1 Electromagnetic Field Tensor $F^{\alpha\beta}$

Tensor is a very sharp thinking tool and can be a very beneficial interactive to mathematics. Much nonsense can be hidden behind a cloud of tensor symbols. Tensor calculation is generally a tedious process and is prone to error when alone by hand. Mastery of mathematical tools cannot be acquired by just reading about them; it is helpful to practice and examine the calculations with handling computations (Yun 2006; Vrbik 2012; Yun 2020a). We used the MATHEMATICA package *Tensorial* 3.0 as a general tensor calculus package (Cabrera 2019). It should be useful both as an introduction to construct the tensor and for calculation with helpful commands, such as DeclarebaseIndices, DefineTensorShortcuts, and SetTensorValuesRules. Some parts of the MATHEMATICA codes in the tensor construction process are presented as *In[#]=:* for the input and *Out[#]=* for the convenience of readers, for the output data to be displayed directly on paper. The initial step is to load "Tensorial'" for the package *Tensorial* 3.0.

In[1]: = Needs["Tensorial'",
   "C:/TensorCalculus3 / Tensorial.m"]
   Clear["Global'*"];

In[2]: = DeclareBaseIndices [{1, 2, 3, 4}]
   DeclareIndexFlavors
   /@{{red, Red},{rocket, SuperStar}};

In[3]: = DefineTensorShortcuts[{{x, A, B, $\mathcal{U}$, E}, 1},
   {F, $\mathcal{F}$, E, B, g, Λ}, 2},{{F, ε}, 4}]
   labs={x, δ, g, Γ};

In[4]: = {Fuu[α β], Euu[α β], Buu[α β], Fuu[α, β]
   Λdd[α, β], Ad[β], xu[α], Ju[α]}

Out[4] = {$F^{\alpha\beta}$, $E^{\alpha\beta}$, $B^{\alpha\beta}$, $F^{\alpha\beta}$, $\Lambda_{\alpha\beta}$, A_β, x^α, J^α}

For the MATHEMATICA coding, input for electric field $E^{\alpha 0}$ and its output are

In[6]: = Euu[α0]==PartialD[labs][Au[0], xd[α]]
   –PartialD[labs][Au[α], xd[0]]

Out[6] = $E^{\alpha 0} = \partial_{x_\alpha} A^0 - \partial_{x_0} A^\alpha$ (21)

In[7]: = % // EinsteinArray[{1, 2, 3}]

Out[7] = {$E^{10} = \partial_{x1}A^0 - \partial_{x0}A^1$, $E^{20} = \partial_{x2}A^0 - \partial_{x0}A^2$, E^{30}
   = $\partial_{x3}A^0 - \partial_{x0}A^3$}

EinsteinArray[1, 2, 3] expresses a tensor with components explicitly.

In[8]: = Buu[$\alpha\beta$]== PartialD[labs][Au[α]], xd[β]]
   – PartialD[labs][Au[β]], xd[α]]

Out[8] = $B^{\alpha\beta} = \partial_{x_\beta} A^\alpha - \partial_{x_\alpha} A^\beta$ (22)

In[9]: = % // EinsteinArray[{1, 2, 3}]

Out[9] = $\begin{pmatrix} B^{11} = 0 & B^{12} = \partial_{x_2}A^1 - \partial_{x_1}A^2 & B^{13} = \partial_{x_3}A^1 - \partial_{x_1}A^3 \\ B^{21} = -\partial_{x_2}A^1 + \partial_{x_1}A^2 & B^{22} = 0 & B^{23} = -\partial_{x_2}A^3 + \partial_{x_3}A^2 \\ B^{31} = -\partial_{x_3}A^1 + \partial_{x_1}A^3 & B^{32} = -\partial_{x_3}A^2 + \partial_{x_1}A^3 & B^{33} = 0 \end{pmatrix}$

In[11]: = Fuu[α, β]==PartialD[labs] Au[β], xd[α]
   – PartialD[labs]Au[α], xd[β]

Out[11] = $F^{\alpha\beta} = -\partial_{x_\beta} A^\alpha + \partial_{x_\alpha} A^\beta$ (23)

For the 4-gradient operators, they are denoted differently in the version of MATHEMATICA as $\left\{ \partial^\alpha = \frac{\partial}{\partial x_\alpha} = \partial_{x_\alpha}, \partial_\alpha = \frac{\partial}{\partial x^\alpha} = \partial_{x^\alpha} \right\}$, we can write ∂^α instead ∂_{x_α} outputs of MATHEMATICA. Then, electric field $\bar{E}$ and magnetic field $\bar{B}$ can be presented with a 4-vector field tensor of rank two as like in Eqs. (21)–(23) could be written in tensor mode as

$$\begin{cases} E^{\alpha 0} = \partial^\alpha A^0 - \partial^0 A^\alpha, \\ B^{\alpha\beta} = -(\partial^\alpha A^\beta - \partial^\beta A^\alpha), \\ F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha. \end{cases} \quad (24)$$

Thus, the components of the vector fields $\bar{E}$ and $\bar{B}$ are presented as the elements of the field tensor $F^{\alpha\beta}$. Now, we can combine the $\bar{E}$ and $\bar{B}$ fields in a tensor $F^{\alpha\beta}$ as follows:

In[12]: = fbRule =
   {Fuu[3,2] → Bd[1], Fuu[2,3] → –Bd[1], Fuu[1,3]
   → Bd[2], Fuu[3,1] → –Bd[2], Fuu[2,1] → Bd[3],
   Fuu[1,2] → –Bd[3], Fuu[0,0] → 0, Fuu[1,1] → 0,
   Fuu[2,2] → 0, Fuu[3,3] → 0}
   feRule =
   {Fuu[1,0] → Ed[1], Fuu[2,0] → Ed[2], Fuu[3,0]
   → Ed[3], Fuu[0,1] → –Ed[1], Fuu[0,2] → –Ed[2],
   Fuu[0,3] → –Ed[3]}

In[13]: = Fuu[α, β]==PartialD[labs][Au[β], xd[α]]
   PartialD[labs][Au[α], xd[β]] Fuu[α, β];

febvalues= %// EinsteinArray[];
febvalue/.fbRule/.feRule;

In[17]: = Fuu[α, β] ==MatrixForm[%%]] == MatrixForm[%]

Out[17] =

$$F^{\alpha\beta} = \partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha}$$

$$= \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad (25)$$

This is a creation of the field tensor $F^{\alpha\beta}$ in MATHEMATICA, which is same to Eq. (20). Thus, we obtain the electromagnetic field of the contravariant field tensor $F^{\alpha\beta}$ in Minkowski space. Then, we can get the covariant field tensor $F_{\alpha\beta}$ from contravariant field tensor $F^{\gamma\eta}$ with metric tensors $g_{\alpha\gamma}g_{\eta\beta}$, for which the MATHEMATICA codes are as follows:

In[18]: =

$$gbasis = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, trans = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, itrans = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In[19]: = SetTensorValueRules[gdd[α, β], gbasis];
  SetTensorValueRules[Fuu[α, β], fbasis];
  TensorValueRules[g];
  TensorValueRules[F];

In[20]: = Clear[F, G, PF, PG, GF, UF, UG];

In[21]: = Fdd[α,β]==gdd[α, γ]gdd[η, β]Fuu[γ, η];
  gdd[α, γ] gdd[η, β]Fuu[γ, η];
  %//MericSmplify[g];
  %%//ToArrayValues[];
  Fdd[α,β]==gdd[α, γ]gdd[η, β]Fuu[γ, η]
  == MatrixForm[%]

Out[25] =

$$F_{\alpha\beta} = g_{\alpha\gamma}F^{\gamma\eta}g_{\eta\beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad (26)$$

Both $F^{\alpha\beta}$ and $F_{\alpha\beta}$ are the *second-rank, antisymmetric field-strength tensors*. Another useful quantity, dual field-strength tensor $\mathcal{F}^{\alpha\beta}$, is obtained from $F_{\gamma\eta}$ with *pseudotensor* $\varepsilon^{\alpha\beta\gamma\delta}$, as shown in Eq. (27).

In[26]: = SetTensorValues[suuuu[χ, λ, μ, ν],
  PermutationPseudotensor[NDim]];
  fdbRule = {Fdd[3,2] → Bd[1], Fdd[2,3] → −Bd[1],
  Fdd[1,3] → Bd[2], Fdd[3,1] → −Bd[2],
  Fdd[2,1] → Bd[3], Fdd[1,2] → −Bd[3],
  Fdd[0,0] → 0, Fdd[1,1] → 0, Fdd[2,2] → 0, Fdd[3,3] → 0};
  fdeRule = {Fdd[1,0]→ −Ed[1], Fdd[2,0]→ −Ed[2],
  Fdd[3,0] → −Ed[3], Fdd[0,1] → Ed[1],
  Fdd[0,2] → Ed[2], Fdd[0,3] → Ed[3]};

In[27]: = $\frac{1}{2}\varepsilon$ uuuu[αβγη] Fdd[γ,η];
  %// ToArrayValues[];
  %/. fdbRule/.fdeRule;
  $\mathcal{F}$uu [α, β]== $\frac{1}{2}\varepsilon$ uuuu[αβγη] Fdd[γ,η] == Matrix[%]

Out[29] =

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}F_{\gamma\eta}\varepsilon^{\alpha\beta\gamma\eta} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix} \quad (27)$$

3.2 Maxwell Equation in Field Tensor $F^{\alpha\beta}$

Inhomogeneous Maxwell equations in free space (in vacuum) are presented in 3-vector differential form as (Jackson 1975)

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho, \quad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \quad (28)$$

We can present this as a covariant field tensor equation with a second-rank field tensor $F^{\alpha\beta}$ and 4-current $J^{\alpha}: J^{\alpha} = (c\rho, \vec{J})$

$$\partial_{x^{\alpha}}F^{\alpha\beta} = \frac{4\pi}{c}J^{\beta} \quad (29)$$

Here we can present it and its components in MATHEMATICA explicitly as:

In[31]: = SetTensorValues[Ju[i], {cρ, j1, j2, j3}];
  PartialD[labs][Fuu[α, β], xu[α]]== (4π)/cJu[β];
  PartialD[labs][Fuu[α, 0], xu[α]]=(4π)/c Ju[0] %
  //ToArrayValues[]

Out[34] = $\partial_{x^{\alpha}}F^{\alpha 0} = \frac{4\pi}{c}J^0$

$$\partial_{x^1}E1 + \partial_{x^2}E2 + \partial_{x^3}E3 = \frac{4\pi J^0}{c}$$

It is just a tensor mode of $\nabla\cdot\vec{E} = 4\pi\rho$ *for* $J^0 = c\rho$. And the case of $\frac{4\pi}{c}J^1$:

In[35]: = PartialD[labs][Fuu[α, 1], xu[α]]=(4π)/c Ju[1]%
//ToArrayValues[]

Out[36] = $\partial_{x^\alpha} F^{\alpha 1} = \frac{4\pi}{c} J^1$

$-\partial_{x^3} B2 + \partial_{x^2} B3 - \partial_{x^0} E1 = \frac{4\pi j1}{c}$

This is the 1st component of $\left(\nabla \times \bar{E} - \frac{1}{c}\frac{\partial \bar{E}}{\partial t}\right)_1 = \left(\frac{4\pi}{c}\bar{J}\right)_1$, its Ampere's law in vector differential mode. The homogeneous Maxwell equations in 3-vector differential mode are

$$\bar{\nabla} \cdot \bar{E} = 0, \ \nabla \times \bar{E} + \frac{1}{c}\frac{\partial \bar{B}}{\partial t} = 0 \qquad (30)$$

It could be presented with the 4-gradient dual field tensor $F^{\alpha\beta}$ as

$$\partial_{x^\alpha} \mathcal{F}^{\alpha\beta} = 0 \qquad (31)$$

Then, we can display the array values of that field tensor equation as

In[38]: = PartialD[labs][$\mathcal{F}$ uu[α, 1], xu[α]] = 0%
//ToArrayValues[]

Out[39] = $\partial_{x^\alpha} \mathcal{F}^{\alpha J} = 0$

$-\partial_{x^0} B_1 + \partial_{x^3} E_2 - \partial_{x^2} E_3 = 0$

It shows that the first component of $\left(\nabla \times \bar{E} + \frac{1}{c}\frac{\partial \bar{B}}{\partial t}\right)_1 = 0$ in field tensor mode is zero.

In[41]: = PartialD[labs][$\mathcal{F}$ uu[α, 0], xu[α]] = 0
% // ToArrayValues[]

Out[42] = $\partial_{x^\alpha} \mathcal{F}^{\alpha 0} = 0$

$\partial_{x^1} B_1 + \partial_{x^2} B_2 - \partial_{x^3} B_3 = 0$

It's $\bar{\nabla} \cdot \bar{B} = 0$. Therefore, the Maxwell equations in inhomogeneous and homogeneous media completely presented in the 4-gradient mode of antisymmetric field strength tensors $F^{\alpha\beta}$ or $\mathcal{F}^{\alpha\beta}$ in MATHEMATICA:

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta, \ \partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \qquad (32)$$

This is the covariant tensor equations of the microscopic Maxwell equations in free space. For the macroscopic Maxwell equations in media, it is necessary to distinguish two field strength tensors, $F^{\alpha\beta} = (\bar{E}, \bar{B})$ and $G^{\alpha\beta} = (\bar{D}, \bar{H})$, where $G^{\alpha\beta}$ is obtained from Eq. (25) by $\bar{E} \to \bar{D}$ and $\bar{B} \to \bar{H}$, while $\bar{D} = \varepsilon \bar{E}$, $\bar{H} = \frac{1}{\mu}\bar{B}$ (Choi 2019). The covariant form of the macro-scopic Maxwell equation is then

$$\partial_\alpha G^{\alpha\beta} = \frac{4\pi}{c} J^\beta, \ \partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \qquad (33)$$

Therefore, the Maxwell equations, in Eqs. (32)–(33), are described with the *second-rank antisymmetric field-strength tensor* of an elegant and complete tensor mode (Arfken 1970; Georgi 1982; Parker 1994).

3.3 Lorentz Transformation of the $F^{\alpha\beta}$ Field Tensor

As the fields $\vec{E}$ and $\vec{B}$ are the elements of a second rank field tensor $F^{\alpha\beta}$, their values in one inertial frame K' can be expressed in terms of the values in another inertial frame K, and, reversely, the values in the K frame in terms of the values in the K' frame according to the Lorentz transformation of $F^{\alpha\beta}$ (Jackson 1975; Parker 1994; Yamamoto 2019):

$$F'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} F^{\alpha\beta} \qquad (34)$$

$$F^{\mu\nu} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} F'^{\alpha\beta} \qquad (35)$$

In the matrix form of Eq. (34) can be written

$$F' = AF\tilde{A} \qquad (36)$$

For the transformation of the rotation and boosting about some axis, A can be written by Lorentz transformation L:

$$\begin{cases} L = -\vec{\omega} \cdot \vec{S} - \vec{\zeta} \cdot \vec{K} \\ A = e^L \end{cases} \qquad (37)$$

where $\bar{\omega}$ and $\bar{\zeta}$ are constant 3-vectors, and they correspond to the rotation and boost of the axis of one coordinate to another. Here, we consider first a simple situation in which $\bar{\omega} = 0$ and $\bar{\zeta} = \zeta\bar{\varepsilon}_1$. Then, $\bar{L} = -\zeta\bar{K}_1$, $A = e^L = (I - K_1^2) - K_1 \sinh\zeta + K_1^2 \cosh\zeta$. The boost vector $\bar{\zeta}$ can be written in terms of the relative velocity $\bar{\beta}$ as $\bar{\zeta} = \hat{\beta} \tanh^{-1}\beta$. Second, in the case of rotation in which $\zeta = 0$ and $\bar{\omega} = \omega\hat{e}_3$, A is determined by $L = -\bar{\omega} \cdot \bar{S}$. Explicitly for the case of boost and rotation, the transforming matrices $A[\zeta\hat{\varepsilon}_1]$ and $A[\omega\hat{e}_3]$ are found to be (Jackson 1975):

$$A\left[\zeta\hat{\varepsilon}_1\right] = \begin{pmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A\left[\omega\hat{\varepsilon}_3\right] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (38)$$

Now, we consider the coordinate transformations of the boost in $\bar{\zeta} = \zeta\,\hat{\varepsilon}_1$ in Minkowski space to visualize the electric fields from the $F'^{\mu\nu}$ and $F^{\mu\nu}$ field tensors, for which MATHEMATICA codes are prepared (In[48]–In[62]). For the macroscopic Maxwell equations, $G^{\alpha\beta} = (\bar{D},\bar{H})$, $G'^{\mu\nu}$ and $G^{\mu\nu}$ are prepared with the MATHEMATICA codes in In[55] and In[62].

$$F'^{\mu\nu} = \Lambda_\alpha^{\,\mu}\Lambda_\beta^{\,\nu}F^{\alpha\beta} \tag{39}$$

$$F^{\mu\nu} = \Gamma_\alpha^{\,\mu}\Gamma_\beta^{\,\nu}F'^{\alpha\beta} \tag{40}$$

where the $\Lambda_\beta^{\alpha} = trans$ and $\Gamma_\beta^{\alpha} = itrans$ tensors described in In[18], $\gamma = (1 - \beta^2)^{-1/2}$, and $\beta = v/c$. Now, we can determine the Lorentz transformations $\{F'^{\mu\nu},\ F^{\mu\nu}\}$ from $\{F^{\alpha\beta},\ F'^{\alpha\beta}\}$ corresponding to a boost along $\bar{\zeta} = \zeta\,\hat{\epsilon}_1$, in Minkowski space by using MATHEMATICA. The complete MATHEMATICA code 1 for evaluation is listed in Appendix.

In[46]: = Clear[F,G,PF,PG,UF,UG,Λ,Γ,primez,unprimez];

In[48]: = PFuu[μ, ν]=Λud[μ, α] Λud[ν, β]Fuu[α, β]
            Λud[μ, α] Λud[ν, β]Fuu[α, β];
            %// ToArrayValues[];

In[55]: = primez%;
            primez//Collect[#,E1] &;
            %//. gRule//Collect[#,γ] &
            PGuu[μ, ν]]=%/. gRule/.mRule//Collect[#,γ] &

In[62]: = UFuu[μ, ν]=Γud[μ, α] Γud[ν, β]PFuu[α, β]
            Γud[μ, α] Γud[ν, β]PFuu[α, β];
            %// MetricSimplify[Γ];
            unprimez=%;
            unprimez// Collect[#,E1] =%;
            UFuu[μ, ν]= %/.gRule/.pRule//Collect[#,γ]&
            UGuu[μ, ν]=%%/. gRule/.qRule//Collect[#,γ]&

Out[59] =

$$F'^{\mu\nu} = \Lambda_\alpha^{\,\mu}\Lambda_\beta^{\,\nu}F^{\alpha\beta} =$$
$$\begin{pmatrix} 0 & -E_1 & -\gamma(E_2 - B_3\beta) & -\gamma(E_3 + B_2\beta) \\ E_1 & 0 & \gamma(E_2\beta - B_3) & \gamma(B_2 + E_3\beta) \\ \gamma(E_2 - B_3\beta) & \gamma(B_3 - E_2\beta) & 0 & -B_1 \\ \gamma(E_3 + B_2\beta) & -\gamma(B_2 + E_3\beta) & B_1 & 0 \end{pmatrix} \tag{41}$$

$$G'^{\mu\nu} = \Lambda_\alpha^{\,\mu}\Lambda_\beta^{\,\nu}G^{\alpha\beta} =$$
$$\begin{pmatrix} 0 & -D_1 & -\gamma(D_2 - H_3\beta) & -\gamma(D_3 + H_2\beta) \\ D_1 & 0 & \gamma(D_2\beta - H_3) & \gamma(H_2 + D_3\beta) \\ \gamma(D_2 - H_3\beta) & \gamma(H_3 - D_2\beta) & 0 & -H_1 \\ \gamma(D_3 + H_2\beta) & -\gamma(H_2 + D_3\beta) & H_1 & 0 \end{pmatrix} \tag{42}$$

Out[69] =

$$F^{\mu\nu} = \Lambda_\alpha^{\,\mu}\Lambda_\beta^{\,\mu}F'^{\alpha\beta} =$$
$$\begin{pmatrix} 0 & -E_1' & -\gamma(E_2' + B_3'\beta) & -\gamma(E_3' - B_2'\beta) \\ E_1' & 0 & -\gamma(E_2'\beta + B_3') & r(B_2' - E_3'\beta) \\ \gamma(E_2' + B_3'\beta) & \gamma(B_3' + E_2'\beta) & 0 & -B_1' \\ \gamma(E_3' - B_2'\beta) & -\gamma(B_2' - E_3'\beta) & B_1' & 0 \end{pmatrix} \tag{43}$$

$$G^{\mu\nu} = \Lambda_\alpha^{\,\mu}\Lambda_\beta^{\,\mu}G'^{\alpha\beta} =$$
$$\begin{pmatrix} 0 & -D_1' & -\gamma(D_2' + H_3'\beta) & -\gamma(D_3' - H_2'\beta) \\ D_1' & 0 & -\gamma(D_2\beta + H_3) & \gamma(H_2' - D_3'\beta) \\ r(D_2' + H_3'\beta) & \gamma(H_3' + D_2'\beta) & 0 & -H_1' \\ r(D_3' - H_2'\beta) & -\gamma(H_2' - D_3'\beta) & H_1' & 0 \end{pmatrix} \tag{44}$$

Thus, we obtain the Lorentz transform of the second-rank antisymmetric field-strength tensor $F^{\alpha\beta}$ in the K system and $F'^{\alpha\beta}$ in the K' system: these are performed in MATHEMATICA platform precisely. Comparing the fields $\{F'^{\mu\nu}, G'^{\mu\nu}\}$ with the fields $\{F^{\alpha\beta}, G^{\alpha\beta}\}$ respectively, we can write down directly the components of the fields $\{\bar{E}', \bar{B}'\}$ in the K' system and of $\{\bar{E}, \bar{B}\}$ in the K system from the Lorentz transformed field tensors in Out[59] and Out[69] as below:

$$\begin{cases} E_1' = E_1 \\ E'_2 = \gamma\left(E_2 - \beta B_3\right) \\ E'_3 = \gamma\left(E_3 + \beta B_2\right) \end{cases} \begin{cases} B'_1 = B_1 \\ B'_2 = \gamma\left(B_2 + \beta E_3\right) \\ B'_3 = \gamma\left(B_3 - \beta E_2\right) \end{cases} \tag{45}$$

$$\begin{cases} E_1 = E'_1 \\ E_2 = \gamma\left(E'_2 + \beta B'_3\right) \\ E_3 = \gamma\left(E'_3 - \beta B'_2\right) \end{cases} \begin{cases} B_1 = B'_1 \\ B_2 = \gamma\left(B'_2 - \beta E'_3\right) \\ B'_3 = \gamma\left(B'_3 + \beta E'_2\right) \end{cases} \tag{46}$$

The components of the Lorentz field tensor may be summarized with three-dimensional vector fields as below (Arfken 1970; Georgi 1982; Parker 1994);

$$\begin{cases} \bar{E}' = \gamma(\bar{E} + \bar{\beta} \times \bar{B}) - \dfrac{\gamma^2}{\gamma + 1}\bar{\beta}(\bar{\beta}\cdot\bar{E}) \\ \bar{B}' = \gamma(\bar{B} - \bar{\beta} \times \bar{E}) - \dfrac{\gamma^2}{\gamma + 1}\bar{\beta}(\bar{\beta}\cdot\bar{B}) \end{cases} \tag{47}$$

$$\begin{cases} \bar{E} = \gamma(\bar{E}' - \bar{\beta} \times \bar{B}') + \dfrac{\gamma^2}{\gamma + 1}\bar{\beta}(\bar{\beta}\cdot\bar{E}') \\ \bar{B} = \gamma(\bar{B}' + \bar{\beta} \times \bar{E}') + \dfrac{\gamma^2}{\gamma + 1}\bar{\beta}(\bar{\beta}\cdot\bar{B}') \end{cases} \tag{48}$$

where $\vec{\beta}$ is the boosting velocity in arbitrary direction in K system. However, the tensor transforms, Eqs. ((41)–(44)) are calculated on the base of boost $\vec{\beta} = \beta\widehat{\varepsilon}_1$ in K system with MATHEMATICA.
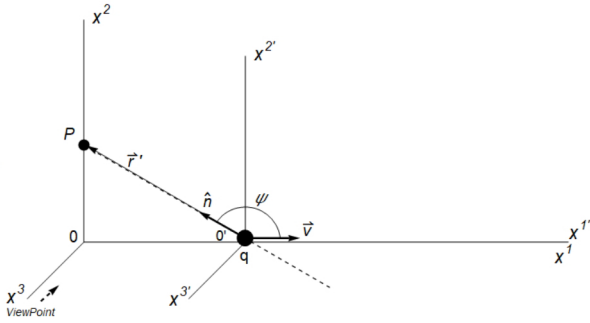
Fig. 1. A particle of charge q at the origin of the K′ system is moving at a constant velocity $\bar{v}$ and passes an observation point P in the K system at an impact parameter b. The lines of the electric field vector for a charged particle at rest and in motion are observed at some viewpoints in the K system.

3.4 Visualizing Electromagnetic Field of a Uniformly Moving Point Charge

As an example of visualizing the transformation of the electromagnetic field tensor, we consider the fields seen by an observer in system K when a point charge q at the origin of the K′ system moves in a straight line of x^1 with a velocity $\bar{v}$ (Yun 2006; Vrbik 2012; Yun 2020a). We suppose that the charge moves in the positive x^1 direction whose closest distance of approach to the observer is b. The observer is at the point P in Fig. 1. At $t = t′ = 0$, the origins of the two coordinate systems coincide, and the charge q is at its closest distance to the observer.

In frame K′, the electric field is calculated as $\vec{E}' = -\vec{\nabla}\left(\frac{q}{r'}\right) = -\left(-\frac{q}{r'^2}\frac{\vec{r}'}{r'}\right) = \frac{q}{r'^3}\left(-vt'\,\hat{e}_1 + b\,\hat{e}_2\right) = \vec{E}'_1 + \vec{E}'_2$ and the magnetic field $\vec{B}' = 0$. Because $t' = rt$ and $r' = \sqrt{b^2 + (vt')^2} = \sqrt{b^2 + (v\gamma t)^2}$, we can write $\vec{E}$ and $\vec{B}$ fields in K system as follows by Eq. (43):

$$
\begin{cases}
E_1 = E'_1 = -\dfrac{q\gamma vt}{\left(b^2 + \gamma^2 v^2 t^2\right)^{\frac{3}{2}}} \\[2ex]
E_2 = \gamma E'_2 = \dfrac{q\gamma b}{\left(b^2 + \gamma^2 v^2 t^2\right)^{\frac{3}{2}}} \\[2ex]
E_3 = 0
\end{cases}
\tag{49}
$$

$$
\begin{cases}
B_1 = B'_1 = 0 \\[1ex]
B_2 = \gamma\left(B'_2 - \beta E'_3\right) = 0 \\[1ex]
B_3 = \gamma\left(B'_3 + \beta E'_2\right) = \beta E_2 = \beta\dfrac{q\gamma b}{\left(b^2 + \gamma^2 v^2 t^2\right)^{\frac{3}{2}}}
\end{cases}
\tag{50}
$$

Here, we write magnetic field in K system as $\vec{B} = \vec{B}_3 = \gamma\left(\vec{\beta} \times \vec{E}'_2\right) = \beta E_2 \hat{e}_3$ in vector form for $B_1 = B_2 = B'_3 = 0$. Then, the fields are observed at point P in K system when the point charge q boosted in the x^1 direction as

$$
\vec{E} = \frac{q\gamma\left(-vt\hat{e}_1 + b\hat{e}_2\right)}{r^3 r^2 \left(1 - \beta^2 \sin^2\psi\right)^{3/2}} = E_1 \hat{e}_1 + E_2 \hat{e}_2
\tag{51}
$$

Whiskbroom Platform of $\vec{E}$ & $\vec{B}$ fields

Field mode   Electromagnetic field

ViewPoint   {0, −0.5, 4.7 }

impact parameter: b   0.1319

boost: β   0.4185

time: t   0.1123

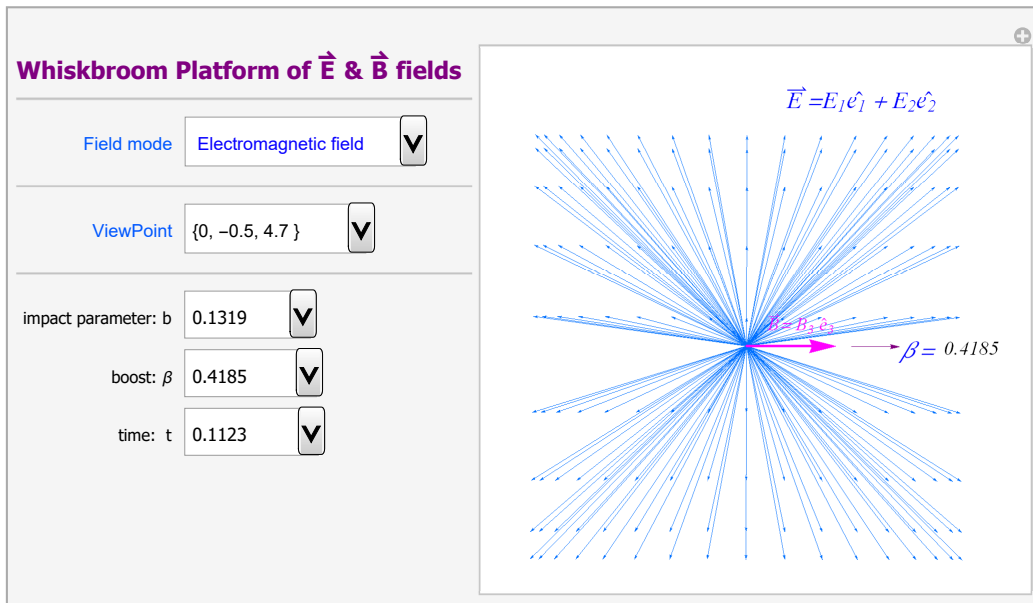$\vec{E} = E_1\hat{e}_1 + E_2\hat{e}_2$

$\beta = 0.4185$

Fig. 2. Whiskbroom platform of $\vec{E}$ and $\vec{B}$ fields. The platform manipulate the electromagnetic fields at P in K system from the charge q at origin in K′ system boosting in direction x^1. It can be possible to manipulate the platform in three field modes: electromagnetic field mode, electric field mode, and magnetic field mode. It present various 'whiskbroom' pattern according to the control parameters (b, β, t, and ViewPoint) in the platform. In the frame K′, the observer point P is at distance $r' = \sqrt{b^2 + (vt')^2}$ away from q. We will to express r′ in terms of the coordinates in K system.
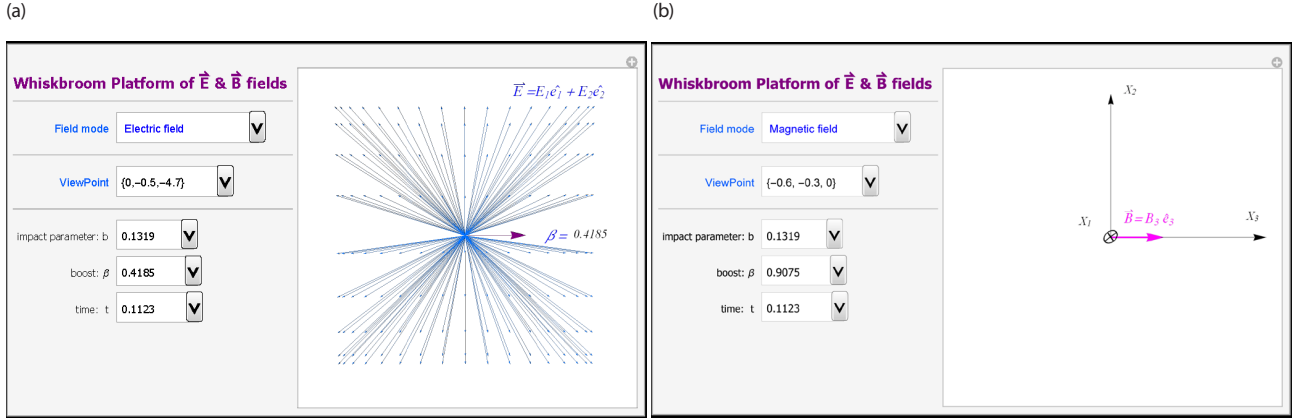
(a) (b)

Fig. 3. Whiskbroom platforms of $\vec{E}$ and $\vec{B}$ fields. (a) In electric field mode, the fields are viewed in take the $\odot x^3$ direction and the boost is on x^1 axis. (b) In magnetic field mode, the fields are viewed along the $\otimes x^1$ direction. The $\vec{B}$ field is along to the x^3 axis.

$$\vec{B} = \gamma\left(\vec{\beta}\times\vec{E}'_2\right) = \beta E_2\hat{e}_3 = B_3\hat{e}_3 \qquad (52)$$

where $\psi = cos^{-1}(\hat{n}\cdot\hat{v})$, and uses the identities: $b = -n\,sin\psi$ and $\gamma^2 - \gamma^2\beta^2 = 1$. We can calculate electric field and magnetic field of a uniformly moving charged particle at P in K using the Eqs. (49–50) and simulate the fields in MATHEMATICA platform as Fig. 2. In this platform, electromagnetic fields can be operated in three field modes: electromagnetic field (Fig. 2), electric field (Fig. 3) and magnetic field modes (Fig. 3). While we control the parameters in $values1=\{b, t, \beta, \psi, \gamma\}$ at the different ViewPoint in the platform, we can observe different field pattern at the some $Viewpoints$ in the platform as shown in Fig. 4.

We can appropriately manipulate the platform for the intended pattern of fields in K system with an adjusted series of control parameters in the MATHEMATICA platform. In the electromagnetic field mode, we can examine the behavior of the magnetic fields, $\gamma\left(\vec{\beta}\times\vec{E}'_2\right) = \beta E_2\hat{e}_3$ while the electric field is $\vec{E} = E_1\hat{e}_1 + E_2\hat{e}_2$ from the graphics in Fig. 5 dynamically, which confirms the electromagnetic interaction of $\vec{B} = \vec{B}_3 = \gamma\left(\vec{\beta}\times\vec{E}'_2\right)$ in Eq. (50). Hence, we can visualize the ap-

parent electric field pattern by changing the boost values (β) of {v, 0.2000, 0.4900, 0.002}, as shown in Fig. 5 in the MATHEMATICA platform. This is so-called $whiskbroom$ pattern of lines of electric field (Arfken 1970; Georgi 1982; Bandyopadhyay 1988).

It is not until we visualize the electric fields in the K frame from the potential of the boosting charge at the K' frame that we can identify clearly the features of the $whiskbroom$ pattern in Minkowski space. The illustrations or descriptions about the $whiskbroom$ pattern in most textbooks have not presented the pattern with graphics yet (Arfken 1970; Georgi 1982; Bandyopadhyay 1988; Fitzpatrick 2014; Yamamoto 2019); the MATHEMATICA platform here can manipulate the lines of $whiskbroom$ pattern precisely by changing the parameters of the control values to the tensor fields. The MATHEMATICA programs for manipulating the field tensor $F^{\alpha\beta}$ and the $whiskbroom$ pattern in this paper are available on the Web site (Yun 2020a; Yun 2020b). Even if MATHEMATICA has not yet been installed on a user's computer, the user may use a CDF file instead (Yun 2020a; Yun 2020b) in CDF Player free download at Wolfram site (Wolfram 2020).
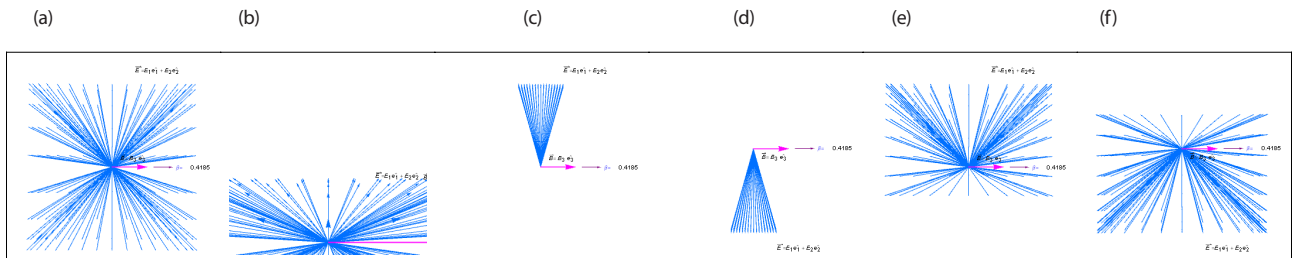
(a) (b) (c) (d) (e) (f)

Fig. 4. Electric fields snapped from the different snap conditions in the $whiskbroom$ platform (Fig. 2): initial setup of $value1=\{c = 1, \gamma = 3, q = 1, \beta = 0.4185, b = 0.1319, t = 0.1123\}$ and $ViewPoint1= \{0.001355, –0.5689, –4.7564\}$. Snaps from the left are: (a) from the initial setup $value1$ and $ViewPoint1$, (b) change $ViewPoint1$ to $ViewPoint3=\{0, –0.1472, 0\}$, (c) change $t \rightarrow 3.1123$ and $ViewPoint2 = \{0.001355, –0.5689, 4.7564\}$, (d) change to $ViewPoint1$ from (c), (e) change $t \rightarrow 1.1312$ from (c), (f) change $t \rightarrow 1.1123$ of $value1$ only.
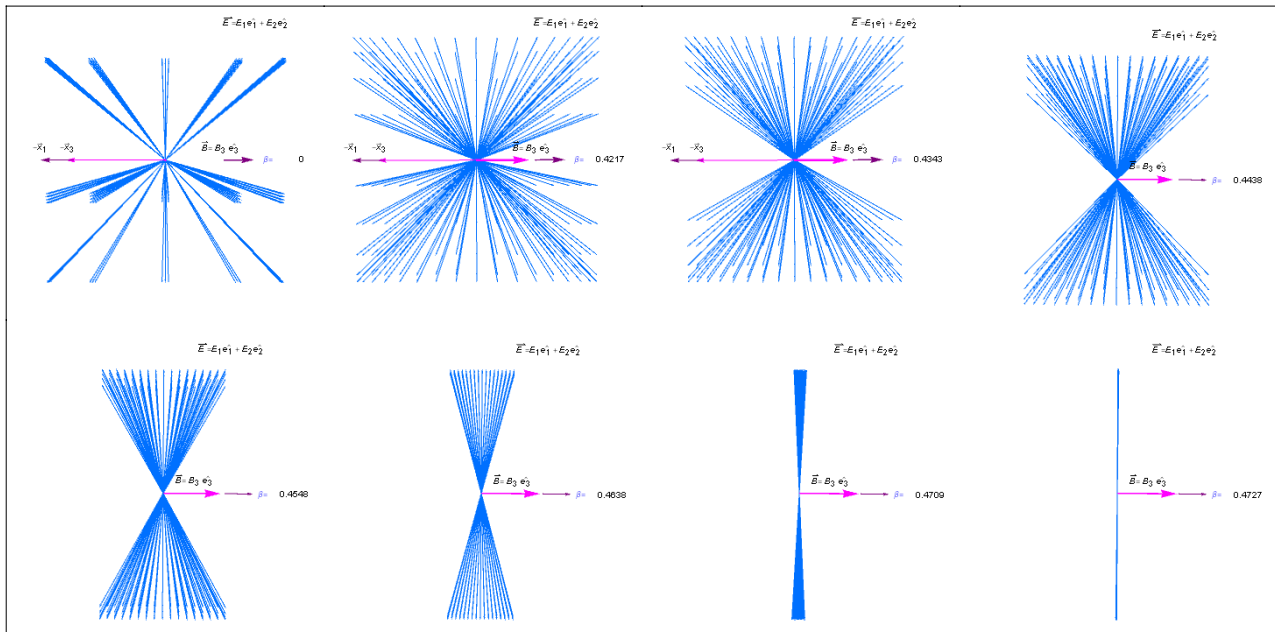
Fig. 5. Snapshots of the *whiskbroom* pattern. The vector field lines of $\vec{E}$ and $\vec{B}$ for a charged particle in motion near $\beta = 0.4100$ along the x^1 axis in the K frame. The patterns are snapped from the *wshiskbroom* platform of electromagnetic field mode on eight β values: $\beta = 0 - 0.4731$ on the same control parameters. $\vec{B}$ field is the $\vec{B} = \beta E_2 \hat{e}_3$ viewed along the $\otimes x^1$ axis while $\vec{E} = E_1 \hat{e}_1 + E_2 \hat{e}_2$ in take the $\odot x^3$ axis in the platform.

4. CONCLUSION

In this paper, we have presented the manifesting of the electromagnetic field tensor $F^{\alpha\beta}$ in Minkowski space, which enables the comprehension and examination of the field tensor with control parameters, such as boosting speed β and impact parameter b, in the MATHEMATICA platform. We can then show the *whiskbroom* pattern precisely in the MATHEMATICA platform. Nowadays, the conceptualization, visualization, and quantization of physics concepts of electromagnetic field tensors has become essential in the development process for knowledge-based information technology. We expect that the MATHEMATICA platform will be a useful starting step for physics students and researchers to construct and examine tensor fields practically in Minkowski space.

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Appendix A: MATHEMATICA CODES

The MATHEMATICA codes below are a part of those codes that produce the tensor fields and plots of the vector fields. However, with only these codes, the vector field could not be produced. The tensor fields are produced with the complete *htensorj* procedure (Yun 2020a; Yun 2020b).

MATHEMATICA code 1

```
In[46]: = Clear[F,G,PF,PG,UF,UG,Λ, Γ,primez,unprimez];
         gRule={ γ² − β²γ² → 1, β²γ² − γ² → −1};
         mRule={E1 → D1, E2 → D2, E3 → D3, B1 → H1,
         B2 → H2, B3 → H3};
         pRule={ E1 → E1', E2 → E2', E3 → E3',B1→
         B1',B2 → B2', B3 → B3'};
         qRule={ E1 → D1', E2 → D2', E3 → D3', B1
         → H1', B2→ H2', B3→ H3'};

In[47]: = SetTensorValueRules[Λud[α, β], trans]
         SetTensorValueRules[Γud[α, β], itrans]
         SetTensorValueRules[Fuu[α, β], fbasis]
         SetTensorValueRules[Guu[α, β], fbasis]
         SetTensorValueRules[Λ]
         SetTensorValueRules[Γ]

In[48]: = PFuu[μ, v]=Λud[μ, α] Λud[v, β]Fuu[α, β]
         Λud[μ, α] Λud[v, β]Fuu[α, β];
         %// ToArrayValues[];

In[55]: = primez=%;
         Primez//Collect[#,E1] &;
         %/.gRule//Collect[#,γ] &
         PGuu[μ, v]]=%/.gRule/.mRule//Collect[#,γ] &

In[62]: = UFuu[μ, v]=Γud[μ, α] Γud[v, β]PFuu[α, β]
         Γud[μ, α] Γud[v, β]PFuu[α, β];
         %// MetricSimplify[Γ];
         unprimez=%;
         unprimez // Collect[#,E1] =%;
         UFuu[μ, v]=%/. gRule/.pRule//Collect[#,γ]&
         UGuu[μ, v]=%%/.gRule/.qRule//Collect[#,γ]&
```

MATHEMATICA code 2

```
In[81]: = Manipulate[
         values1={c → 1, γ → 3, q → 1, b → 0.131943, t →
         0.11233, ψ → 1.6648};
         view = {{0.001355, −0.56895, −4.7654}, {0.001355,
         0.56895, 4.7654}, {0, −0.1472,0}, {0, 0.1319, 0},
         {0, −0.1319, 0}, {0, −1319.1, 1319.1}};
         tensor2 = Evaluate[{Sin[−q(b² + v²t²)^(1/2)x)/(b² +
         (vt)²)^(3/2))γ²(1 − v²Sin[ψ]²)^(3/2)], Cos[q(b²(b² + v²t²)^(1/2)y)/
         (b² + (vt)²)^(3/2))γ²(1 − v²Sin[ψ]²)^(3/2)]}//.values1];
         tensor3 = Re[Evaluate[Insert[tensor2,0,3]]];
         efields = {{Thickness[0.021175], Arrowheads
         [0.0625], Hue[0.5937], Table[Arrow[[{{0,0,0},
         tensor3}], {x, −π, π, π/8},{y, −π, π, π/8}],
         ControllerLinking→ All}}};
         gr50 = {{Text[StyleForm["β=",{0.65, −0.01,0}]]}};
         gr51 = {{Text[StyleForm["v",v,{0.85, −0.01,0}]]}};
         boot1 = {Thickness[0.00453], Arrowheads
         [0.02788], Purple, Arrow[{{0,0,0}, {0.51, −0.02,0}}]],
         AspectRatio→Automatic};
         p1 = Graphics3D[Join[{efields, boost1, gr50, gr51}],
         Axes → False, Boxed → False,
         PlotRange → {{−1,1},{−1,1},{−1,1.3}},
         ImageSize →{380,380},
         ViewPoint → view[[p]],
         DisplayFunction → $DisplayFunction;
         Show[ p1],
         Style["Whiskbroom pattern", Bold],
         {b, 0.1319, Row [{"impact parameter:",
         Subscript["b", Style["",Italic]]}},
         {{1.0 × 10⁻¹⁵, 0.01, 0.1219, 0.1319, 0.4713, 0.6714,
         0.8319, 1.2221}},
         {{v, 0.4185, Row[{"boost :", Subscript["β",
         Style["", Italic]]}},
         {{1.0 × 10⁻¹⁵, 0.01, 0.4185, 0.4217, 0.4343, 0.4709,
         0.4727, 0.4731}},
         {{t, 0.1123, Row [{"time :", Subscript["t", Style["°", Italic]]}},
         {{1.0 × 10⁻¹⁵, 0.1123, 0.3123, 1.1003, 3.1123}},
         {{p, 1, Row[{"view point: ",Subscript["p",
         Style[" ",Italic]]}},
         {{1, 2, 3, 4, 5, 6}},
         TrackedSymbols: → {b,v,t,p},
         Initialization: → (huefunc[x] :=
         Hue[If[v >1,0.,If[x<0,0.671,0.671(1-x)]]])]

Out[81] = Fig. 2
```